Fourier Series, Integrals and Transforms

Definition: Fourier Series are infinite series designed to represent general, periodic functions interms of Simple ones, namely cosines and sines. They constitute a very important tool, in particular in solving problems that involve ODEs, and PDEs.

Thus, Fourier series are the basic tool for representing periodic functions, which play an important role in applications. A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x and if there is some positive number p, called a period of $f(x)$ such that $f(x+p) = f(x)$. Familiar periodic functions are the cosine and sine functions [Examples of functions that are not periodic are $x, x^2, e^x, \cosh x, \ln x$).

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- Fourier Series of } f(x)$$

with Fourier Coefficients:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

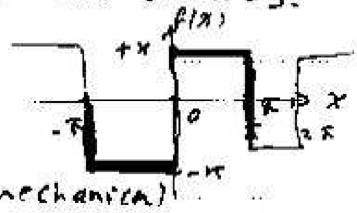
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad n = 1, 2, \ldots$$

Example:- Periodic rectangular wave:

Find the Fourier Coefficients of the periodic function $f(x)$ given by:

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x)$$



Functions of this kind occur as external forces action on mechanical Systems, electromotive forces in electrical circuits, etc.

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$ [This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$, 2π is zero]

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\cos nx\, dx + \int_{0}^{\pi} K\cos nx\, dx\right]$$

$$a_n = \frac{1}{\pi}\left[-K\frac{\sin nx}{n}\Big|_{-\pi}^{0} + K\frac{\sin nx}{n}\Big|_{0}^{\pi}\right] = 0$$

because $\sin nx = 0$ at $-\pi, 0, \pi$ for all $n = 1, 2, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\sin nx\, dx + \int_{0}^{\pi} K\sin nx\, dx\right]$$

$$b_n = \frac{1}{\pi}\left[K\frac{\cos nx}{n}\Big|_{-\pi}^{0} - K\frac{\cos nx}{n}\Big|_{0}^{\pi}\right]$$

Since $\cos(-\alpha) = \cos\alpha$, and $\cos(0) = 1$, this gives

$$b_n = \frac{K}{n\pi}\left[\cos(0) - \cos(-n\pi) - \cos(n\pi) + \cos(0)\right] = \frac{2K}{n\pi}\left(1 - \cos(n\pi)\right)$$

Now, $\cos(\pi) = -1$, $\cos(2\pi) = 1$, $\cos(3\pi) = -1$, etc. in general

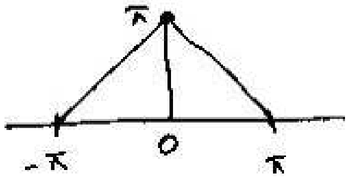
$$\cos(n\pi) = \begin{bmatrix} -1 & \text{for odd } n \\ 1 & \text{for even } n \end{bmatrix} \quad \text{and thus} \quad 1 - \cos(n\pi) = \begin{bmatrix} 2 & \text{for odd } n \\ 0 & \text{for even } n \end{bmatrix}$$

hence the Fourier Coefficients b_n of our functions are:

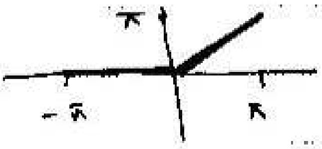
$$b_1 = \frac{4K}{\pi},\quad b_2 = 0,\quad b_3 = \frac{4K}{3\pi},\quad b_4 = 0,\quad b_5 = \frac{4K}{5\pi},\ \ldots$$

Thus Fourier Series for the given $f(x) = \frac{4K}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

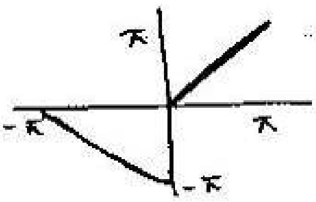
Ex: Find Fourier Transform for the following Periodic functions:

1. 

Ans. $\left[\frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)\right]$

2. 

Ans. $\left[\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right) + \left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)\right]$

3. 

Ans. $\left[-\frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right) + 2\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)\right]$